Comment on "The Cluster Expansion for the Self-Gravitating gas and the Thermodynamic Limit", by de Vega and Sánchez (astro-ph/0307318)

V. Laliena*
Departamento de Física Teórica, Universidad de Zaragoza, C. Pedro Cerbuna 12, E-50009 Zaragoza (Spain)
(Dated: September 8, 2003)

In a series of papers, de Vega and Sánchez claimed that the thermodynamic limit of a self-gravitating system can be taken by letting the number of particles, N, and the volume, V, tend to infinity keeping the ratio $N/V^{1/3}$ constant [1]. This limit, which I call *diluted* following the terminology of the first paper of [1], is different from the usual thermodynamic limit, where the density N/V is kept constant. The relevant variable for the diluted limit, which can be found by naive dimensional analysis, is $\eta = Gm^2N/V^{1/3}T$.

Recently, I proved rigorously that the diluted limit does not give a well defined thermodynamic limit [2]: the relevant thermodynamic potentials are not extensive and the thermodynamic quantities suffer from the same problems as in the usual thermodynamic limit. For instance, the free energy scales with $N^{5/3}$. However, in "The Cluster Expansion for the Self-Gravitating gas and the Thermodynamic Limit" [3], de Vega and Sánchez continue arguing that the diluted limit gives extensive thermodynamic potentials and well behaved thermodynamic quantities at sufficiently high temperature, i.e. for $\eta > \eta_c$, where η_c depends on the thermodynamic ensemble as well as on the geometry of the system boundary. To mantain these statements, these authors try to show that "the statements made in ref.[7] [reference [2] of the present paper] have crucial failures which invalidate the conclusions given in ref.[7]". However, the argument they give is based on a misunderstunding of the proof of nonexistence of the diluted limit given in [2], and is easily refuted, as will be seen in the following.

Let us briefly remember the proof of nonexistence of the diluted limit given in [2]. Consider a system of N classical particles, enclosed on a region of linear size R (so that $V = R^3$) interacting via a gravitational potential conveniently regularized at short distances. In the diluted limit we have $N \to \infty$, $R \to \infty$, with N/R constant. The variable η of ref. [3] is, by definition,

$$\eta = \frac{Gm^2N}{RT}\,, \tag{1}$$

since $V = R^3$ is the volume of the region available for the system. Now, let us consider a region of linear size R_0, with $N \sim R_0^3$, enclosed in the available space of the system. Note that $R_0 \ll R$. Using a simple sequence of inequalities, it is proven in [2] that

$$\mathcal{Z}_{\rm C} \geq \frac{R_0^{3N}}{N!}\,\exp[\beta N(N-1)\kappa/R_0]\,, \tag{2}$$

where $\mathcal{Z}_{\rm C}$ is the canonical partition function, $\beta = 1/T$ is the inverse of the temperature, and $\kappa > 0$ is Gm^2 times a geometrical number independent of R_0 if R_0 is large. The above inequality shows that the free energy grows with N at least as $N^{5/3}$, and therefore is not extensive.

The authors of [3] argue that since to derive inequality (2) I have introduced another length R_0, the relevant variable is $\eta = Gm^2N/TR_0$, and, since $R_0 \sim N^{1/3}$, we have $\eta \gg \eta_c$, so that the system is deep in the collapse phase, where the results of [1, 3] do not apply.

Obviously, this is wrong. R_0 is not a characteristic length of the system. It is an auxiliary mathematical length, introduced just to prove that the diluted limit is ill-behaved. It has no physical meaning, and hence it is left unspecified. It can have any value, the only retriction being that it must scale with the number of particles as $R_0 \sim N^{1/3}$. By definition, the length entering η in equation (1) of ref. [3] is the linear size of the spatial region available to the system. Hence, $\eta = Gm^2/TR$. Only in such case the integrals over the coordinates that appear in equations (2.9), (2.15), etc., of [3] can be taken between 0 and 1. Inequality (2) can be written as

$$\mathcal{Z}_{\rm C} \geq \frac{R_0^{3N}}{N!}\,\exp[\eta(N-1)\bar{\kappa}R/R_0]\,, \tag{3}$$

where $\bar{\kappa} > 0$ is now a dimensionless purely geometrical number independent of the size of the system. Since $R/R_0 \sim N^{2/3}$, we see that the free energy scales $N^{5/3}$ if η is kept constant, and therefore is nonextensive.

It is clear that the grand canonical partition function cannot scale as $\exp[Ng(\eta,\mu)]$ if the canonical partition function grows with N faster than $\exp[Nf(\eta)]$. Hence, the grand canonical ensemble cannot give an extensive thermodynamic potential, either.

Inequalities (2) or (3) are valid whatever the value of the temperature (i.e., of η). Hence, they prove that the gas phase obtained in refs. [1, 3] does not exist in the diluted limit. For a finite system, we expect a gas phase at high temperature (small η) and a collapse phase at low temperature (large η), with the two phases separated by a phase transition or crossover at some η_c. As the system size grows, η_c will decrease towards zero, the gas phase will shrink and the collapse phase will eventually cover the whole phase diagram.

*Electronic address: laliena@posta.unizar.es

In [3] it is shown that the diluted limit exists order by order in the cluster expansion (a similar proof was given in [2] for the high temperature expansion). This contradicts the proof of nonexistence of the diluted limit. Since inequalities (2) or (3) are derived rigorously, with no assumption, the high temperature and cluster expansions cannot be valid. The reasons why this kind of expansions fail have been analyzed in [2]. Basically, there are two possibilities:

i) The series in η/N may not converge in the $N \to \infty$ limit, due to the contribution of high order diagrams that are naively suppressed by powers of $1/N$, but which actually may give a significant contribution ought to the short distance divergences (the cut-off behaves as $a = A/N$, where A is a fixed length, *Cf.* Eq. (2.8) of the paper). Concerning this point, the statement that appears at the end of the introduction of the paper [3], "one can take the limit $N \to \infty$ and **then** $a \to 0$" does probably not hold due to the singularities of the integrals that give the coefficients of the cluster expansion. The rigorous limit is, obviously $N \to \infty$, $a \to 0$, with Na fixed. The modification of the procedure to take the $N \to \infty$ limit may be at the core of the failure of the cluster expansion developed in [3].

ii) Even if the cluster expansion were convergent, the series could not represent the thermodynamic potential, since it is in contradiction with the rigorous result of [2]. In deriving the cluster expansion there is at least one mathematically unjustified exchange of limits that may invalidate the equality between the thermodynamic potential and the cluster series. The cluster expansion (I use the notation of [3]),

$$Q_N(\eta) = 1 + \sum \int f_{ij} + \sum \int f_{ij} f_{kl} + \ldots\,, \qquad (4)$$

is rigorous, since the number of terms in the sums is finite for finite N. To proceed further, the authors of [3] expand f_{ij} in powers of η/N, and exchange the sum and integral. Mathematically, it can be very difficult to analyze the conditions under which this exchange of limits is allowed, but we can get some insight from physical intuition. Inequality (3) suggests that the system is collapsed for large N if η is of order 1 respect to $1/N$. This means that the cluster expansion is dominated by the higher order terms (many particles within a cluster). Hence, the canonical (Gibbs) integration measure on configuration space is very concentrated on collapsed configurations. Hence, this measure is very different from the flat measure $\prod_i d^3 r_i$ that correspond to a gas. Keeping only the dominant of the expansion of f_{ij} in powers of η/N means that one is using effectively the flat measure. To recover something similar to the concentrated true measure, one has to sum an infinite number of terms in η/N. In other words, the expansion in η/N is valid only in the gas phase. Inequality (3) suggests that the gas phase can only take place for η of the order of $N^{-2/3}$. Hence, the radius of convergence of the expansion in η shrinks to zero as $N \to \infty$.

Similar statements claiming the existence of the diluted thermodynamic limit have been made in [4] without even mentioning the results of ref. [2].

[1] H.J. de Vega and N. Sánchez, Phys. Lett. B490, 180 (2000); Nucl. Phys. B625, 409 (2002); *ibid*, 460.
[2] V. Laliena, Nucl. Phys. B668, 403 (2003) (astro-ph/0303301).
[3] H.J. de Vega and N.G. Sánchez, astro-ph/0307318.
[4] H.J. de Vega and J.A. Siebert, astro-ph/0305322.